MATHEMATICS OF COMPUTATION, VOLUME 28, NUMBER 125, JANUARY, 1974

A Note on Congruent Numbers

By Ronald Alter and Thaddeus B. Curtz

Abstract. An integer a is called a congruent number if and only if there are positive integer solutions to the system of equations

$$x^2 + ay^2 = z^2 \quad \text{and} \quad x^2 - ay^2 = t^2.$$

In this note congruent numbers are discussed and a table of known square-free congruent numbers less than 1000 is exhibited.

1. Introduction. An integer a is called a congruent number if and only if there exist positive integer solutions to the rationalized system of Diophantine equations

$$x^2 + ay^2 = z^2 \quad \text{and} \quad x^2 - ay^2 = t^2. \tag{1}$$

It is easy to see that cn^2 is a congruent number if and only if c is a congruent number. Thus it suffices to study only square-free congruent numbers.

The earliest reference to congruent numbers in the literature dates back to a tenth century Arab manuscript. For a more detailed account of the early history of congruent numbers, see Dickson [3, Chapter 16, pp. 459–472]. More recent references to congruent numbers can be found in several texts. See, for example, Uspensky and Heaslet [6] and Mordell [5]. For the most recent results on congruent numbers, see Alter, Curtz and Kubota [1].

It is known that solving Eq. (1) is equivalent to solving the single Diophantine equation

$$x^4 - a^2y^4 = z^2. \tag{2}$$

Also, every congruent number a satisfying Eq. (1) must be of the form

$$uv(u^2 - v^2) = aw^2. \tag{3}$$

There are other forms that a congruent number may take; in fact, the following are all congruent numbers:

$$x^4 + 4y^4, \qquad 2x^4 + 2y^4, \qquad x^4 - y^4. \tag{4}$$

If x and y have opposite parity, then the following are also congruent numbers:

$$x^4 + 6x^2y^2 + y^4, \qquad x^4 - 6x^2y^2 + y^4. \tag{5}$$

In Alter, Curtz and Kubota [1], other results on congruent numbers and a list of known results on noncongruent numbers is given.

In 1915, Gérardin [4] listed 62 square-free congruent numbers less than 1000 for which x is less than 3722 in Eq. (1). (The list actually contains 63 numbers but one is

Received January 26, 1973.

AMS (MOS) subject classifications (1970). Primary 65A05; Secondary 10B05.

Key words and phrases. Congruent numbers, Diophantine equation.

not square-free and thus is an error.) At the time he posed the problem of finding all square-free congruent numbers less than 1000. This was followed by a note of Bastien [2] in which he listed all square-free congruent numbers less than 100. Using new results and a computer search based primarily on Eq. (3), Alter, Curtz and Kubota [1] determine 334 of the 608 square-free numbers less than 1000 to be either congruent (198 numbers) or noncongruent (136 numbers). These numbers are exhibited in tables of congruent and noncongruent numbers.

It is the aim of this note to find new congruent numbers less than 1000. This is done by using a computer search based on (4) and (5), in which the parameters were taken as large as double precision would allow. A discussion of this program can be found in [1]. In fact only 18 new congruent numbers were found, leaving 256 square-free numbers less than 1000 that are still undetermined.

A complete list of known (square-free) congruent numbers less than 1000 is contained below in Table 1.

TABLE 1

Square-Free Congruent Numbers < 1000

5	6	7	13	14	15	21	22	23	29
30	31	34	37	38	39	41	46	47	53
55	61	62	65	69	70	71	77	78	79
85	86	87	93	94	95	101	102	109	110
111	118	119	134	137	138	141	142	143	145
149	151	154	158	159	161	165	166	174	181
182	190	194	205	206	210	214	215	219	221
226	231	239	246	254	255	257	262	265	278
285	286	287	291	299	302	310	313	318	319
323	326	330	334	349	353	357	358	366	371
382	386	390	391	395	398	399	410	422	426
429	434	438	442	445	446	454	455	457	462
465	470	478	479	502	505	509	510	511	514
517	518	526	527	533	535	542	546	561	565
566	574	582	583	602	609	614	615	622	629
645	646	651	658	662	663	669	670	671	674
689	694	709	710	718	719	721	731	734	741
751	758	759	761	766	777	791	793	798	799
805	806	813	814	821	838	862	866	870	878
879	886	889	890	897	901	903	905	910	915
926	934	935	943	949	951	957	958	959	966
974	982	985	987	995	998				

These new congruent numbers further strengthen the following conjecture which also appears in [1].

Conjecture. *If* $n \equiv 5, 6$ *or* 7 (mod 8) *then* n *is a congruent number.*

The first three such n still unsettled are 103, 127, 133.

Although this note deals primarily with congruent numbers, our previously published list of known noncongruent numbers is repeated here for the convenience of the reader. For more information about Table 2, the reader is again referred to [1].

TABLE 2

Noncongruent Numbers < 1000

1	2	3	10	11	17	19	26	33	35
42	43	51	57	58	59	66	67	73	74
82	83	89	91	97	106	107	114	122	129
130	131	139	146	163	170	177	178	179	186
193	201	202	209	211	218	227	233	241	249
251	274	281	283	290	298	307	314	321	331
346	347	362	370	379	393	394	401	417	419
433	443	449	457	458	466	467	473	489	491
499	523	530	537	538	547	554	562	563	571
586	587	601	610	617	619	626	633	634	641
643	649	659	673	681	683	691	698	737	739
746	753	754	769	778	787	794	811	817	818
827	842	849	859	883	907	913	914	921	922
929	937	947	962	971	993				

The author would like to thank R. Waldner for programming assistance.

Department of Computer Science
University of Kentucky
Lexington, Kentucky 40506

1. R. ALTER, T. B. CURTZ & K. K. KUBOTA, *Remarks and Results on Congruent Numbers*, Proc. Third Southeastern Conference on Combinatorics, Graph Theory and Computing, 1972, pp. 27–35.

2. L. BASTIEN, "Nombres congruents," *Intermédiare des Math.*, v. 22, 1915, pp. 231–232.

3. L. E. DICKSON, *History of the Theory of Numbers*. Vol. II, Carnegie Institute of Washington, 1920.

4. A. GÉRARDIN, "Nombres congruents," *Intermédiare des Math.*, v. 22, 1915, pp. 52–53.

5. L. J. MORDELL, *Diophantine Equations*, Pure and Appl. Math., vol. 30, Academic Press, New York, 1969. MR **40** #2600.

6. J. V. USPENSKY & M. A. HEASLET, *Elementary Number Theory*, McGraw-Hill, New York, 1939. MR **1**, 38.